Utility Functions
Part II - The Logrithmic Utility Function

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In this white paper we will define the logrithmic utility function. To that end we will work through the following hypothetical problem from Part I...

Our Hypothetical Problem

We are given the following investment payoffs and market assumptions...

Table 1: Investment Payoffs

Symbol	Payoff	Probability
W_1	200	0.40
W_2	500	0.30
W_3	700	0.20
W_4	950	0.10

Table 2: Market Assumptions

Symbol	Description	Value
μ	Annual risk-free interest rate (%)	4.50
κ	Annual expected investment return (%)	8.00
T	Investment term in years (#)	3.00

Questions:
1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?
2. What is the value of the scalar alpha?
3. Graph utility, marginal utility, and risk aversion.

Expected Wealth

We will define the variable p_i to be the probability of realizing wealth W_i (the i'th random investment payoff). The equation for expected wealth is... [2]

$$\mathbb{E}\Big[W\Big] = \sum_{i=1}^{n} p_i\, W_i \tag{1}$$

Using Equation (1) above and the data in Table 1 above, the expected increase in wealth from our investment as defined in Table 1 above is...

$$\mathbb{E}\Big[W\Big] = 200 \times 0.40 + 500 \times 0.30 + 700 \times 0.20 + 950 \times 0.10 = 465.00 \tag{2}$$

Logrithmic Utility Function

We will define the variable α to be a scalar whose value is greater than zero. The equation for our logrithmic untility function and it's first and second derivatives are the following equations...

$$U(W_i) = \ln(1 + \alpha\, W_i) \text{ ...where... } U'(W_i) = \alpha\,(1 + \alpha\, W_i)^{-1} \text{ ...and... } U''(W_i) = -\alpha^2\,(1 + \alpha\, W_i)^{-2} \tag{3}$$

Using Equation (3) above, we can make the following statements...

$$U(W_i) = 0 \text{ ...when... } W_i = 0 \text{ ...and... } U(W_i) > 0 \text{ ...when... } W_i > 0 \tag{4}$$

The equation for our initial guess of the value of scalar parameter α where W_{max} is the maximum investment payoff is...

$$\text{if... } \ln(1+\alpha\,\text{maximum wealth}) = 1.00 \text{ ...then... } \alpha = \left(\text{Exp}\left\{1.00\right\} - 1.00\right) \Big/ \text{maximum wealth} \tag{5}$$

We defined the variable λ to be the Arrow-Pratt measure of risk aversion. Using Equation (3) above, the equation for the measre of risk aversion is... [2]

$$\lambda = -\frac{U''(W_i)}{U'(W_i)} = -\frac{-\alpha^2\,(1+\alpha\,W_i)^{-2}}{\alpha\,(1+\alpha\,W_i)^{-1}} = \alpha\,(1+\alpha\,W_i)^{-1} \tag{6}$$

Note that the value of λ in Equation (5) above implies decreasing absolute risk aversion. Investors will invest a greater percentage of their wealth in risky assets as they get wealthier

Expected Utility

Using Equation (3) above, the equation for the expected utility of wealth is... [2]

$$\mathbb{E}\Big[U(W)\Big] = \sum_i^n p_i\,U(W_i) = \sum_i^n p_i\,\ln(1+\alpha\,W_i) \tag{7}$$

Using Equation (7) above, the equation for the derivative of the expected utility of wealth with respect to the scalar parameter α is...

$$\frac{\delta}{\delta\,\alpha}\,\mathbb{E}\Big[U(W)\Big] = \sum_{i=1}^n p_i\,W_i\,(1+\alpha\,W_i)^{-1} \tag{8}$$

Certainty Equivalent

We will define the variable CE to be the value of the certainty equivalent at time T. The dollar value of the certainty equivalent is such that the utility of the certainty equivalent is equal to the utility of expected wealth. This statement in equation form is... [2]

$$U(CE) = \mathbb{E}\Big[U(W)\Big] \tag{9}$$

Using Equations (3) and (7) above, we can rewrite Equation (9) above as...

$$\ln(1+\alpha\,CE) = \sum_i^n p_i\,\ln(1+\alpha\,W_i) \tag{10}$$

In Table 2 above we defined the variable μ to be the risk-free rate and the variable κ to be the risk-adjusted discount rate. If we are given the discount rate κ then using Equation (1) above the equation for the value of the certainty equivalent at time T is...

$$\text{if... } \kappa = \left(\mathbb{E}\Big[W\Big] \Big/ CE\,(1+\mu)^{-T}\right)^{1/T} - 1 \text{ ...then... } CE = \mathbb{E}\Big[W\Big]\left(\frac{1+\mu}{1+\kappa}\right)^T \tag{11}$$

Given the value of the certainty equivalent at time T (Equation (11) above) we want to solve Equation (10) above for the value of the scalar α. We will make the following scalar parameter definitions...

$$\alpha = \text{Actual value of the scalar alpha ...and... } \hat{\alpha} = \text{Guess value of the scalar alpha} \tag{12}$$

Using Equations (10) and (12) above, we will define the function $f(\alpha)$ to be...

$$f(\alpha) = \sum_i^n p_i\,\ln(1+\alpha\,W_i) - \ln(1+\alpha\,CE) = 0 \tag{13}$$

Using Equations (10) and (12) above, we will define the function $f(\hat{\alpha})$ to be...

$$f(\hat{\alpha}) = \sum_i^n p_i\,\ln(1+\hat{\alpha}\,W_i) - \ln(1+\hat{\alpha}\,CE) \tag{14}$$

Using Equation (8) above, the derivative of Equation (14) above with respect to the scalar α is...

$$f'(\hat{\alpha}) = \sum_{i=1}^{n} p_i \, W_i \, (1 + \alpha \, W_i)^{-1} - CE \, (1 + \hat{\alpha} \, CE)^{-1} \tag{15}$$

To solve for κ, the Newton-Raphson equation that we will iterate is... [?]

$$\alpha + \hat{\epsilon} = \hat{\alpha} + \frac{f(\alpha) - f(\hat{\alpha})}{f'(\hat{\alpha})} \, \Bigg| \, f(\alpha) = \text{Equation (13)}, \; f(\hat{\alpha}) = \text{Equation (14)}, \; f'(\hat{\alpha}) = \text{Equation (15)} \tag{16}$$

Answers To Our Hypothetical Problem

1. Given a discount rate of 8.00%, what is the value of the certainty equivalent at time T and at time zero?

Using Equations (2) and (11) above and the data in Table 1 above, the value of the certainty equivalent at time T is...

$$CE = 465.00 \times \left(\frac{1 + 0.0450}{1 + 0.8000} \right)^3 = 421.24 \tag{17}$$

Using Equation (18) above and the data in Table 1 above, the value of the certainty equivalent at time zero is...

$$PVCE = 421.24 \times (1 + 0.0450)^{-3} = 369.13 \tag{18}$$

2. What is the value of the scalar alpha?

Using Equation (5) above and the data in Table 1 above, the guess value of our scalar is...

$$\alpha = \left(\text{Exp}\left\{ 1.00 \right\} - 1.00 \right) \Big/ 950.00 = 0.001809 \tag{19}$$

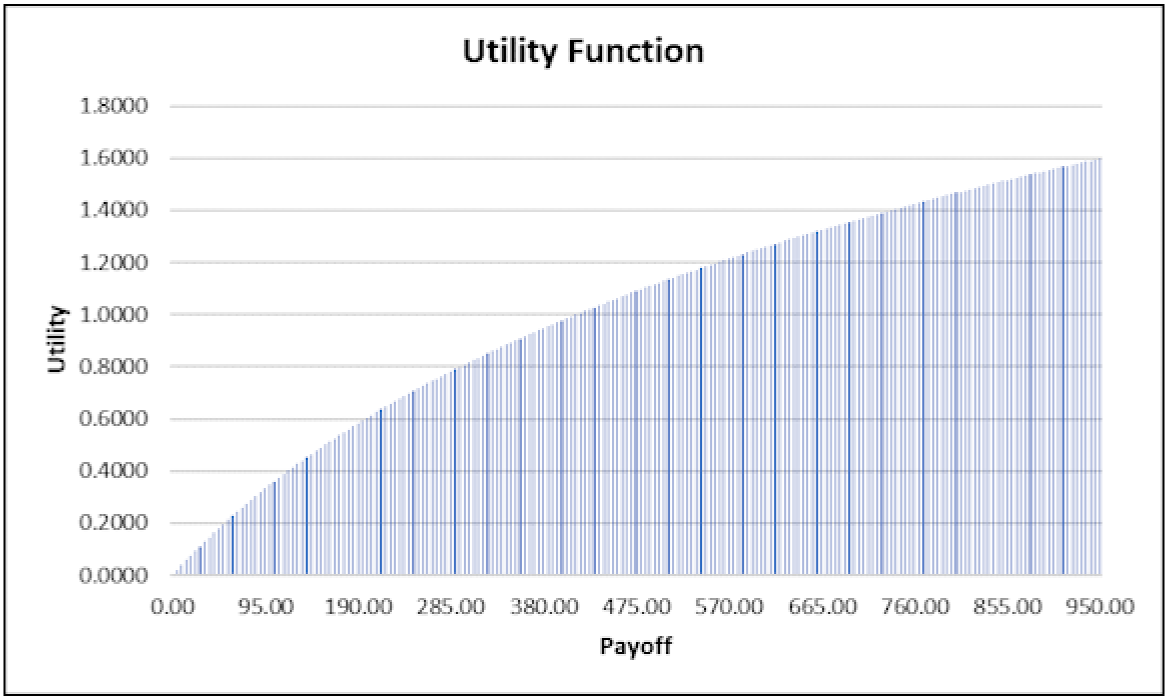
Using Equations (16) and (17) above and the data in Table 1 above, the value of our scalar is...

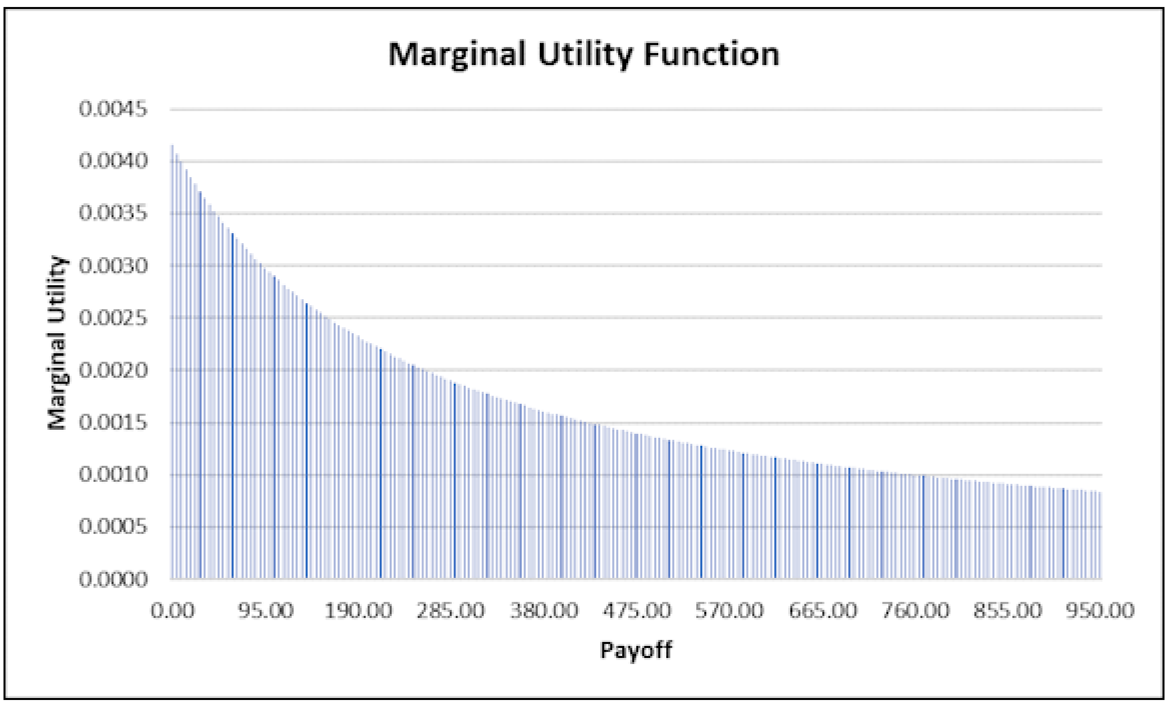
Iteration	Scalar	$f(\alpha)$	$f(\hat{\alpha})$	$f'(\hat{\alpha})$
0	0.001809	0.000000	0.013966	-4.839
1	0.004695	0.000000	-0.002983	-5.337
2	0.004136	0.000000	0.000109	-5.722
3	0.004155	0.000000	0.000000	-5.709
4	0.004155	0.000000	0.000000	-5.709
5	0.004155	0.000000	0.000000	-5.709

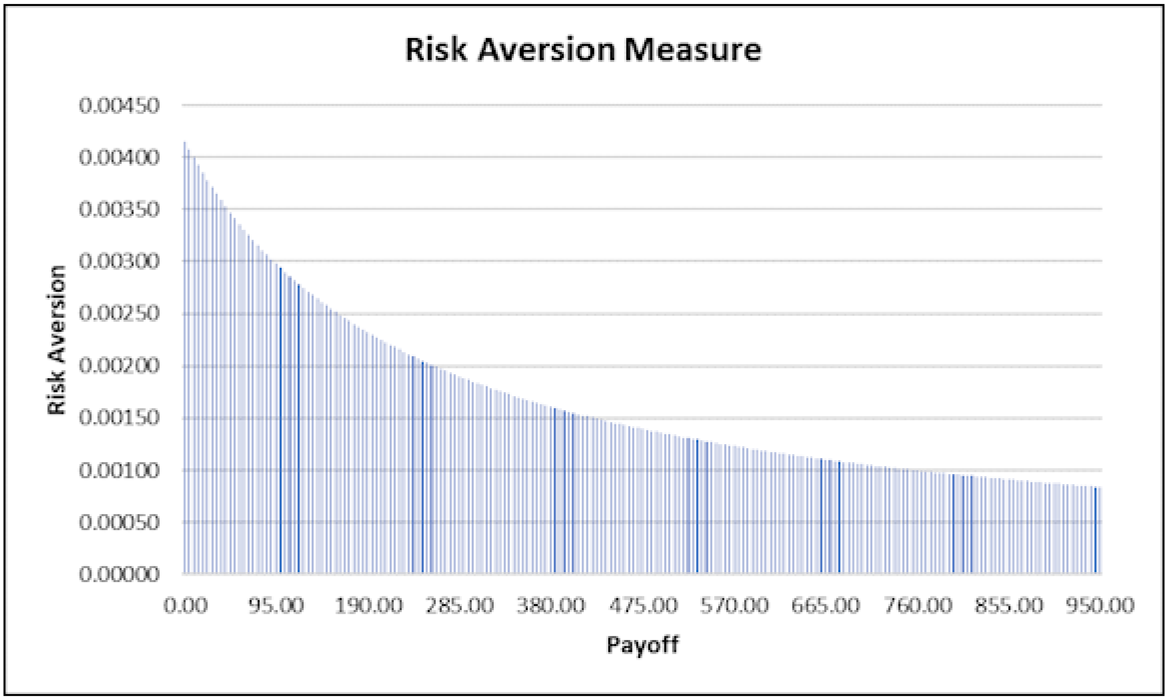
Using the results in the table above, the answer to the question is...

$$\text{if...} \;\; \alpha = \text{scalar value} = 0.004155 \;\; \text{...then...} \;\; \kappa = \text{investment rate of return} = 8.00\% \tag{20}$$

3. Graph utility, marginal utility, and risk aversion.







References

[1] Gary Schurman, *Introduction To Utility Funtions*, October, 2023.

[2] Gary Schurman, *Introduction To Utility Funtions*, October, 2023.